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EXPONENTIAL STABILIZATION OF NON-LINEAR STOCHASTIC SYSTEMS*

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We consider the stabilization of non-linear systems whose parameters are subjected to "white noise". For stochastic systems with non-linear feedback, we derive sufficient (frequency-domain) conditions of exponential stabilization by a controller that uses information about the system output (incomplete system state information). The problem of the stabilization of linear stochastic systems has been studied in some detail /1-3/. Yet for non-linear stochastic systems we only have general theorems that reduce the stabilization problem to finding a stochastic Lyapunov function /4, 5/.

In this paper, we derive sufficient conditions of exponential stabilization by methods of the theory of absolute stochastic stability. The advantages of these methods are well-known: the specific Lyapunov function is not required, and its existence in the class of functions "quadratic form plus integrals over non-linearities" is easily checked /6/. The latest results of this theory for stochastic systems /7/ make it possible to solve the stabilization problem for a wide class of non-linear systems with parametric disturbances.

1. Formulation of the problem.

We consider a controllable dynamic system described by Ito's differential equation

$$x^{\cdot} = (A_0 + \Sigma A_j w_j^{\cdot}) x + (b_0 + \Sigma b_j w_j^{\cdot}) u + (q_0 + \Sigma q_j w_j^{\cdot}) \varphi(\sigma), \quad \sigma = \nu^* x \tag{1.1}$$

Here x is the n-dimensional state vector, u is the d-dimensional control vector, σ is the l-dimensional vector of observed variables, φ is the m-dimensional vector function describing the non-linear feedback or allowing for other non-linear effects in the system, A_j, b_j, q_j ($j = 0, 1, \ldots, s$) are appropriately dimensioned constant matrices, and w_j ($j = 1, \ldots, s$) are independent standard Wiener processes; here and henceforth, summation is over j from $j = 1$ to $j = s$, unless otherwise stated.

The class of admissible non-linear functions $\varphi(\sigma)$ is described in accordance with the general theory of absolute stability /6/. Let

$$F_1(\sigma, x, \varphi, \psi) = \sigma^* r\sigma + 2\sigma^* p\varphi + \varphi^* g\varphi - \Sigma f_j^* \theta\psi \tag{1.2}$$
$$f_j = \operatorname{diag}[\Lambda_j \Lambda_j^*], \quad \Lambda_j = \nu^*(A_j x + q_j\varphi), \quad j = 1, \ldots, s$$

The symbol $\operatorname{diag}[\cdot]$ is the vector formed from the main diagonal elements of the matrix in brackets: f_j ($j = 1, \ldots, s$) are l-dimensional vectors, and ψ is an m-dimensional vector. The real matrices $r = r^*, p, g = g^*, \theta$ are $l \times l, l \times m, m \times m, l \times m$ respectively. We assume that the matrix θ satisfies the following conditions: a) it is non-zero only when $\nu^* b_j = 0$ for all $j = 1, \ldots, s$; b) if condition a) holds, then the element θ_{kl} of the matrix θ may be non-zero only if φ_l is a continuously differentiable function of a single variable σ_k the k-th component of the vector σ.

We assume that the non-linearity φ satisfies the condition

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$$F_1(\sigma, x, \varphi(\sigma), \quad \varphi'(\sigma)) \leqslant 0, \quad \sigma = \nu^* x, \quad \varphi(0) = 0 \tag{1.3}$$

($\varphi'(\sigma)$ is the vector whose i-th component is equal to $d\varphi_i/d\sigma_k$ if $\theta_{ki} \neq 0$ or 0 if $\theta_{ki} = 0$). Moreover, let the $(m \times l)$ matrix μ be such that if $\theta_{ki} \neq 0$, then $\mu_{ij} = 0$ for $j \neq k, 1 \leqslant j \leqslant l$, and let the linear function $\mu\sigma$ satisfy the inequality (1.3) and the function φ have the property

$$2\theta_{ki}\int_0^\sigma \varphi_i(\xi)\,d\xi \leqslant \theta_{ki}\mu_{ik}\sigma^2, \quad \sigma \in R^1 \tag{1.4}$$

The class of admissible non-linearities consists of all the continuous functions $\varphi(\sigma)$ that satisfy (1.3) and (1.4).

We consider the problem of exponential stabilization (in mean square /1-4/ or with probability 1 /4, 5/) of the non-linear system (1.1) with admissible non-linearity φ when the information about the state x is incomplete and only the output σ is observable.

In what follows, we solve this problem using the controller

$$z^\cdot = (A_0 + q_0\mu\nu^*)\,z + b_0 u + C(\sigma - \nu^* z), \quad u = K^* z \tag{1.5}$$

(z is a n-dimensional vector and K, C are constant $n \times d$ and $n \times l$ matrices, respectively).

The contoller (1.5) was used in /1, 2/ to stabilize the linear system (we call it system A) obtained from (1.1) by substituting $\varphi = \mu\sigma$. It clearly belongs to the class of systems considered in this paper. The assumption that the controller (1.5) exponentially stabilizes the system A in mean square is central to our analysis. This assumption can be checked for given K, C using the results of /1, 2/, which also enable us to choose "optimal" values of these parameters. The system A is essentially a linear comparison system for the class of non-linear system (1.1).

2. Sufficient conditions of stabilization. We will now give the general conditions when the controller (1.5) exponentially stabilizes system (1.1).

Let $\varepsilon = x - z$. Subtracting the first relation in (1.5) term by term from (1.1), substituting the second relation from (1.5), and setting $z = x - \varepsilon$, we obtain an equation in the unknown ε.

$$\begin{aligned} \varepsilon^\cdot = (A_0 + q_0\mu\nu^* - C\nu^* - \Sigma b_j K^* w_j^\cdot)\,\varepsilon - q_0\mu\sigma + \\ (\Sigma(A_j + b_j K^*)\,w_j^\cdot)\,x + (q_0 + \Sigma q_j w_j^\cdot)\,\varphi(\sigma) \end{aligned} \tag{2.1}$$

Eqs.(1.1), (2.1) form a closed non-linear system of stochastic equations of the form

$$\begin{aligned} & y^\cdot = (B_0 + \Sigma B_j w_j^\cdot)\,y + (D_0 + \Sigma D_j w_j^\cdot)\,\varphi(\sigma), \quad \sigma = \eta^* y \\ & y = \mathrm{col}\,(x_1, \ldots, x_n\ \varepsilon_1, \ldots, \varepsilon_n) \\ & B_0 = B_0(\mu) = \begin{Vmatrix} A_0 + b_0 K^* & -b_0 K^* \\ -q_0\mu\nu^* & A_0 + q_0\mu\nu^* - C\nu^* \end{Vmatrix}, \quad D_0 = \begin{Vmatrix} q_0 \\ q_0 \end{Vmatrix} \\ & \eta = \begin{Vmatrix} \nu \\ 0 \end{Vmatrix}, \quad B_j = \begin{Vmatrix} A_j + b_j K^* & -b_j K^* \\ A_j + b_j K^* & -b_j K^* \end{Vmatrix}, \quad D_j = \begin{Vmatrix} q_j \\ q_j \end{Vmatrix} \\ & j = 1, \ldots, s \end{aligned} \tag{2.2}$$

Exponential stabilization of system (1.1) by the controller (1.5) is equivalent to global stability of the equilibrium of the Ito Eq.(2.2). The theory of global absolute stability for equations of the form (2.2) was developed in /7/*. Theorem 1 below follows from this theory as it applies to our particular problem. In order to state the theorem, we will need the following notation .

$$\begin{aligned} & R = \eta r \eta^*, \quad Q = \eta p + B_0^* \eta\theta, \quad G = g + \theta^*\eta^* D_0 + D_0^*\eta\theta \\ & F(y, v) = y^* R y + 2y^* Q v + v^* G v, \quad y \in R^{2n}, \quad v \in R^m \end{aligned} \tag{2.3}$$

Let $y(t)$ be a path of the equation

$$y^\cdot = (B_0 + \Sigma B_j w_j^\cdot)\,y + (D_0 + \Sigma D_j w_j^\cdot)\,v(t), \quad y(0) = 0 \tag{2.4}$$

which is obtained from (2.2) by cutting the non-linear feedback loop $v = \varphi(\sigma)$. As a result, we obtain a linear system (2.4) controlled by the stochastic process $v(t)$. The set of admissible controls V consists of $v(\cdot)$ satisfying the following conditions:

a) the process $v(t)$ and the corresponding solution of problem (2.4), $y(t)$, are consistent with the stream of σ-algebras induced by the Wiener processes $w_j(t)$ $(j = 1, \ldots, s)$, $t \geqslant 0$ /4/;

*See also Brusin V.A. and Ugrinovskii V.A., Stability of stationary motions of systems with parametric disturbances, Preprint 220, Gor'kii: NIRFI (1986).

b) the following inequalities hold (E is the expectation symbol):

$$\int_0^\infty E|v(t)|^2 dt < +\infty, \quad \int_0^\infty E|y(t)|^2 dt < +\infty$$

Theorem 1. Assume that the controller (1.5) mean-square exponentially stabilizes the linear system A and, moreover, there is $\varepsilon > 0$ such that

$$E\int_0^\infty F(y(t), v(t))\,dt \geqslant \varepsilon\int_0^\infty E|v(t)|^2 dt, \quad \forall v \in V \tag{2.5}$$

Then this controller also mean-square exponentially stabilizes system (1.1).

This assertion follows directly from the exponential stability theorem (see the work cited in the footnote), which claims that under our assumptions the function

$$y^*Hy + \sum_{i,k} 2\theta_{ik}\int_0^{\sigma_k} \varphi_i(\xi)\,d\xi$$

is a stochastic Lyapunov function for Eq.(2.2). The existence of a Lyapunov function of this form also ensures exponential stabilization of (1.1) with probability 1 (see Theorem 5.8.1 /4/):

$$|x(t)| \leqslant \text{const}\cdot e^{-\gamma t}, \quad |\varepsilon(t)| \leqslant \text{const}\cdot e^{-\gamma t} \quad (\exists\gamma > 0) \tag{2.6}$$

The constant in (2.6) is almost surely finite and is determined by the initial value $x(0)$, $z(0)$. (Lyapunov-type theorems of stabilization with probability 1 are also given in /5/).

Theorem 1 covers a wide class of controllable dynamic systems with parametric disturbance. As we show below, special estimates of the integral on the left-hand side of (2.5) can be used to obtain frequency-domain stabilization conditions from Theorem 1.

3. Stabilization of a system with interference in the feedback loop.

Consider a system of the form

$$x^{\cdot} = A_0x + b_0u + (q_0 + q_1w_1^{\cdot})\,\varphi(\sigma), \quad \sigma = \nu^*x \tag{3.1}$$

$$0 \leqslant \varphi(\sigma)\,\sigma \leqslant h\sigma^2, \quad \varphi' \geqslant 0 \tag{3.2}$$

(the output σ and the function φ are both scalar, $l = m = 1$).

The frequency-domain corollary of Theorem 1 is obtained in three stages. In the first stage, we choose a linear comparison system of the form of the system A with its stabilizing controller (1.5). As the comparison system, we take the linear system obtained from (3.1) by substituting $\varphi = h\sigma$ (this function satisfies conditions (3.2)),

$$x^{\cdot} = (A_0 + q_0h\nu^* + q_1h\nu^*w_1^{\cdot})\,x + bu, \quad \sigma = \nu^*x \tag{3.3}$$

The stabilizability of this system by the controller (1.5) (with the parameter μ equal to h) is equivalent to exponential stability of the trivial solution of the linear equation

$$y^{\cdot} = Py + hD_1\sigma w_1^{\cdot}, \quad \sigma = \eta^*y, \quad P = B_0(h) + h\,D_0\eta^* \tag{3.4}$$

An algebraic criterion of mean-square stability for Eq.(3.4) is given in /8/. The controller parameters K, C should be chosen from this criterion. The matrix P, and also $A_0 + q_0h\nu^* + b_0K^*$, $A_0 + q_0h\nu^* - C\nu^*$ are necessarily Hurwitz matrices. Let K, C be such that $B_0(h)$ is also a Hurwitz matrix. As we will see below, these assumptions add nothing new to our previous assumptions: the location of the spectrum of the matrix B_0 to the left of the imaginary axis is a necessary condition for stabilization by the controller (1.5) of system (3.1) with $\varphi \equiv 0$ contained in the class of systems that we seek to stabilize.

In the second stage, we describe the class of admissible non-linearities (3.2) by inequalities of the form (1.3), (1.4). Any function φ satisfying (3.2) obviously satisfies the inequalities

$$\frac{1}{h}\varphi^2 - \sigma\varphi - \theta(\nu^*q_1)^2\varphi^2\varphi' \leqslant 0, \quad \theta\int_0^\sigma \varphi(\xi)\,d\xi \leqslant \frac{\theta h\sigma^2}{2} \tag{3.5}$$

for any $\theta \geqslant 0$.

We can now formulate a frequency-domain condition which ensures that inequality (2.5) holds. This is the third and final stage in deriving the "frequency-domain" stabilization theorem.

Let

$$\varkappa_j(\lambda) = (\lambda I - B_0)^{-1} D_j, \quad \chi_j(\lambda) = \eta^* \varkappa_j(\lambda), \quad j = 0, 1$$

Theorem 2. Assume that the triple of matrices (A_0, b_0, ν) is controllable and observable; the parameter μ is equal to h, and the matrices K, C are chosen so that the controller (1.5) exponentially stabilizes the linear system (3.3) in mean square; the eigenvalues of the matrix B_0 for these parameter values lie to the left of the imaginary axis. For some $\theta > 0, \varepsilon > 0$, let

$$1/h - 2\beta^{1/2} - \mathrm{Re}\,(1 - 2i\omega\theta)\,\chi_0(i\omega) \geqslant \varepsilon, \quad \forall \omega \in (-\infty, +\infty) \tag{3.6}$$

(the number β is defined by (3.7) below).

Then the controller (1.5) with these μ, K, C exponentially stabilizes system (3.1) (in mean square and with probability 1 in the sense of (2.6)) with any function $\varphi(\sigma)$ satisfying condition (3.2).

Proof. If we cut the non-linear feedback loop in system (3.1), (2.1), we obtain a linear controllable dynamic system of the form (2.5).

$$y^{\cdot} = B_0 y + D_0 v + D_1 v w_1^{\cdot}, \quad v = v(t), \quad y(0) = 0, \quad \sigma = \eta^* y(t)$$

By assumption, if $v \in V$, then $E\,|v|^2, E\,|y(t)|^2, E\,|\sigma(t)|^2$ are Lebesgue integrable on $[0, +\infty)$. Let $v(t), y(t)$, and $\sigma(t)$ vanish for $t < 0$. Then the Fourier-Laplace transforms of these processes are related by the identities

$$y_F(i\omega) = \varkappa_0(i\omega) v_F(i\omega) + \varkappa_1(i\omega) J(i\omega), \quad \sigma_F(i\omega) = \chi_0(i\omega) v_F(i\omega) + \chi_1(i\omega) J(i\omega)$$

$$J(i\omega) = \int_0^\infty v(t)\, e^{-i\omega t}\, dw_1(t)$$

(the subscript F denotes the Fourier transform of the corresponding process). These identities lead to

$$y(t) = y_0(t) + y_1(t)$$

$$y_0(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{Re}\,\varkappa_0(i\omega)\, v_F(i\omega)\, d\omega, \quad y_1(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{Re}\,\varkappa_1(i\omega)\, J(i\omega)\, d\omega$$

By (3.5), the coefficients of the quadratic form F (2.3) are given by

$$R = 0, \quad Q = -\frac{1}{2}\eta + \theta B_0 \eta, \quad G = \frac{1}{h} + 2\theta(\nu^* q_0)$$

Let

$$\beta = \frac{1}{2\pi} \int_{-\infty}^{\infty} |Q^* \varkappa_1(i\omega)|^2\, d\omega \tag{3.7}$$

The integral (3.7) is finite, since the eigenvalues of the matrix B_0 lie to the left of the imaginary axis. From the above representation of $y(t)$ and the inequality $2y_1^* Q v \geqslant -\beta^{-1/2} |Q y_1|^2 - \beta^{1/2} |v|^2$, we obtain the bound

$$\begin{aligned} \int_0^\infty EF(y(t), v(t))\, dt \geqslant \int_0^\infty EF(y_0(t), v(t))\, dt - \\ \beta^{-1/2} \Big\{ \int_0^\infty E\,|Q y_1(t)|^2\, dt + \beta \int_0^\infty E\,|v(t)|^2\, dt \Big\} = \\ \int_0^\infty E\,\{F(y_0(t), v(t)) - 2\beta^{1/2} |v|^2\}\, dt \end{aligned} \tag{3.8}$$

(The last equality is based on the properties of Ito integrals). Fourier transforming the right-hand side of (3.8), using Parseval's equality and inequality (3.6), we obtain the theorem. The theorem is proved.

Theorem 2 applies to monotone arbitrarily rapidly increasing non-linearities. The case $\varphi' \leqslant d$ is covered by Theorem 3 (see below), and positivity of φ' is not required.

Now let $\mu = 0$. The value of the matrix B_0 and $\varkappa_j(\lambda)$, $\chi_j(\lambda)$ $(j = 0, 1)$, β are changed accordingly.

Theorem 3. Assume that the triple of matrices (A_0, b_0, ν) is controllable and observable; the parameters in the controller (1.5) have been chosen so that $\mu = 0$, and the eigenvalues of the matrices $A_0 + b_0K^*$, $A_0 - C\nu^*$ lie in the complex plane to the left of the imaginary axis. For some $\theta \leqslant 0$, $\varepsilon > 0$, let

$$1/h + \theta(\nu^* q_1)^2 d - 2\beta^{1/2} - \operatorname{Re}(1 - 2i\omega\theta)\chi_0(i\omega) \geqslant \varepsilon, \ \forall\omega \in R^1 \tag{3.9}$$

Under these conditions, the controller (1.5) exponentially stabilizes system (3.1).

Conditions (3.6), (3.9) are analogous to Popov's frequency-domain condition from the theory of absolute stability and has the same geometrical interpretation /6/: the hodograph of the modified frequency characteristic $X = -\operatorname{Re}\chi_0(i\omega)$, $Y = -\omega \operatorname{Im}\chi_0(i\omega)$ lies strictly to the right of the Popov line $1/h_1 + X + 2\theta Y = 0$. The value of h_1 is obtained from Theorem 2.3:

$1/h_1 = 1/h - 2\beta^{1/2}$ for condition (3.6)

$1/h_1 = 1/h + \theta(\nu^* q_1)^2 d - 2\beta^{1/2}$ for condition (3.9)

This interpretation may be useful in many applications, e.g., when studying the robustness of the stabilizability properties to interference (robustness estimates of the stabilization system).

Example. It is required to stabilize the system

$$\begin{gathered}\sigma'' + 2\alpha\sigma' + \sigma = \eta,\ \eta' + \beta\eta = \varphi(\sigma)(1 + qw') + u \\ q > 0,\ -1 < \alpha < 0,\ \beta > 0,\ h < -2\alpha(1 + 2\alpha\beta + \beta^2) < \beta\end{gathered} \tag{3.10}$$

with the function φ satisfying inequalities (3.2).

Given this relationship of the parameters in the uncontrolled system $(u \equiv 0)$, all the linear systems of this class prove to be unstable. Using Popov's frequency-domain criterion /6/, we can show that without interference $(q = 0)$, the stabilizing controller for (3.10), (3.2) has the form

$$\begin{gathered}\sigma_1' = \rho_1 + c(\sigma - \sigma_1),\ \rho_1' + 2\alpha\rho_1 + \sigma_1 = \eta_1 \\ \eta_1' + \beta\eta_1 = h\sigma_1 + u,\ u = k\sigma_1\end{gathered} \tag{3.11}$$

with appropriately chosen k, c. Thus, for $\alpha = -0.2$, $\beta = 2$, $h = 0.2$ we may take $k = 1.681$, $c = 2$. Analysis of the closed-loop system (3.10), (3.11) based on Theorem 2 shows that for $q < q^\circ \approx 0.77$ the controller (3.11) also exponentially stabilizes system (3.10) both in mean square and with probability 1, i.e., the controller (3.11) is robust to "white noise" of intensity not exceeding q°. The values of the parameters θ, h_1 corresponding to this threshold intensity which ensure that the frequency-domain condition (3.6) holds are $\theta = 0.6$, $h_1 \approx 0.264$.

4. Stabilization of a system with interference in the system and the control channel. Let us derive a frequency-domain corollary of Theorem 1 for Ito systems of the form

$$x' = (A_0 + A_1 w_1')x + (b_0 + b_2 w_2')u + q_0\varphi(\sigma),\ \sigma = \nu^* x \tag{4.1}$$

with scalar σ and φ; w_1, w_2 are independent scalar standard Wiener processes. We assume that $\nu^* b_2 = 0$ and that the non-linear function $\varphi(\sigma)$ has the properties

$$h_1\sigma^2 \leqslant \varphi(\sigma)\sigma \leqslant h_2\sigma^2,\quad \varphi' \geqslant 0,\quad \int_0^\sigma \varphi(\xi)\,d\xi \leqslant \frac{\mu\sigma^2}{2},\quad \mu \in [h_1, h_2] \tag{4.2}$$

Clearly, every function φ satisfying these constraints is contained in the set of functions defined by the inequality

$$(\varphi - h_1\sigma)(\varphi - h_2\sigma) - \theta(\nu^* A_1 x)^2\varphi' \leqslant 0,\ \theta \geqslant 0 \tag{4.3}$$

Following the procedure of Sects.1-3, we take the linear comparison system in the form

$$x' = (A_\mu + A_1 w_1')x + (b_0 + b_2 w_2')u,\ A_\mu = A_0 + q_0\mu\nu^* \tag{4.4}$$

which is obtained from (4.1) by substituting $\varphi = \mu\sigma$. This substitution satisfies the conditions (4.3). The parameters of the controller (1.5) stabilizing system (4.4) are chosen following the suggestions of /1, 2/.

In order to derive the frequency-domain corollary of Theorem 1, we will need an auxiliary lemma.

Let

$$\begin{gathered}P = B_0(\mu) + \mu D_0\eta^*,\ R_1 = (h_1 - \mu)(h_2 - \mu) - \theta\mu B_1^*\eta\eta^* B_1 \\ Q_1 = [-(h_1 + h_2)/2 + \mu]\eta + \theta P^*\eta,\ G_1 = 1 - \varepsilon + 2\theta(\nu^* q_0), \\ 0 < \varepsilon \leqslant 1\end{gathered} \tag{4.5}$$

Let $\|\cdot\|$ be the spectral matrix norm in the space of $(2n \times 2n)$ matrices induced by the Euclidian norm in R^{2n} /9/, and let the spectrum of the matrix P lie in the region $\mathrm{Re}\,\lambda < -\alpha$, $\alpha > 0$ and $\|\exp(Pt)\| < \rho \exp(-\alpha t)$, $\rho > 0$.

We know /9/ that the matrix equations

$$P^*H_0 + H_0P = \gamma^{-1}Q_1Q_1^*, \ \gamma > 0 \tag{4.6}$$

$$P^*T_0 + T_0P = -B_1^*H_0B_1 - B_2^*H_0B_2 - R_1 \tag{4.7}$$

$$P^*T_k + T_kP = -B_1^*T_{k-1}B_1 - B_2^*T_{k-1}B_2, \ k \geqslant 1$$

have unique symmetric solutions H_0, T_k, $k \geqslant 0$, where $H_0 \leqslant 0$ since $Q_1Q_1^* \geqslant 0$ and $T_k \leqslant 0$ since $R_1 \leqslant 0$.

Lemma. Let $\beta = \rho^2(\|B_1\|^2 + \|B_2\|^2)/2\alpha$. Under the above assumptions, the equation

$$P^*H + HP + \delta(B_1^*HB_1 + B_2HB_2) = -B_1^*H_0B_1 - B_2^*H_0B_2 - R_1 \tag{4.8}$$

has a unique solution $H = H^* \leqslant 0$ which is representable in series form as

$$H = T_0 + T_1\delta + T_2\delta^2 + \ldots \tag{4.9}$$

The series is convergent in the circle $|\delta| < 1/\beta$.

Proof. The fact that the series (4.9) satisfies (4.8) can be checked by direct substitution. Convergence of the series (4.9) in any norm is equivalent to the convergence of the numerical series

$$\langle y, T_0y\rangle + \langle y, T_1y\rangle\delta + \langle y, T_2y\rangle\delta^2 + \ldots$$

We may take $|y| = 1$. The radius of convergence of the last series is $r = 1/\lim|\langle y, T_ky\rangle|^{1/k}$ as $k \to +\infty$. By the bound $|y^*T_ky| \leqslant \beta^k\|T_0\|$, we have $r \geqslant 1/\beta$, which it was required to prove.

Now let H be the solution of Eq.(4.8), $M = -P^*H - HP$, $M \leqslant 0$, $\varkappa_0(\lambda) = (\lambda I - P)^{-1}D_0$, $\chi(\lambda) = \eta^*\varkappa_0(\lambda)$, $\varkappa_j(\lambda) = (\lambda I - P)^{-1}B_j$ $(j = 1, 2)$.

Theorem 4. Let the triple of matrices (A_μ, b_0, ν) be controllable and observable; let the parameters of the controller (1.5) be such that it stabilizes the system (4.4). Let β, γ be the numbers from the lemma; $\gamma_1 > 0$ and $1 + 1/\gamma_1 < 1/\beta$; $\varepsilon > 0$ is such that

$$1 - \gamma - (1 + \gamma_1)\,\mathrm{Re}\,\varkappa_0^*(i\omega)M\varkappa_0(i\omega) - \mathrm{Re}\,(h_1 + h_2 - 2\mu - 2i\omega\theta)\chi(i\omega) \geqslant \varepsilon, \ \forall\omega \in (-\infty, +\infty) \tag{4.10}$$

Under these conditions, the controller (1.5) stabilizes the non-linear system (4.1) with an arbitrary non-linearity satisfying (4.2).

Proof. A necessary condition for stabilization of system (4.4) by the controller (1.5) is that the eigenvalues of the matrix P (4.5) are located so that $\mathrm{Re}\,\lambda_i \leqslant -\alpha$ $(\exists\alpha > 0)$. Then for some $\rho > 0$, $\|\exp(Pt)\| \leqslant \rho\exp(-\alpha t)$.

Substituting $v_1 = v - \mu\sigma$, we reduce Eq.(2.4) to the form

$$y^{\cdot} = Py + D_0v_1 + (B_1w_1^{\cdot} + B_2w_2^{\cdot})y, \ \sigma = \eta^*y, \ y(0) = 0$$

Processes satisfying this equation have Fourier-Laplace transforms related by the identity (since $v_1 \in V$)

$$y_F(i\omega) = \varkappa_0(i\omega)v_{1F}(i\omega) + \sum_{j=1}^{2}\varkappa_j(i\omega)\int_0^\infty y(t)e^{-i\omega t}dw_j(t)$$

By the assumptions of the theorem, for $\delta = 1 + 1/\gamma_1$, the solution of Eq.(4.8) exists and the series (4.9) converges. Generalizing the technique of /10/, we can show that

$$\int_0^\infty Ey^*My\,dt \geqslant \frac{1+\gamma_1}{2\pi}\int_{-\infty}^{\infty}\varkappa_0^*(i\omega)M\varkappa_0(i\omega)E|v_{1F}(i\omega)|^2d\omega + \left(1 + \frac{1}{\gamma_1}\right)\int_0^\infty Ey^*(B_1^*HB_1 + B_2HB_2)y\,dt$$

$$\int_0^\infty E2y^*Q_1v_1\,dt \geqslant -\gamma\int_0^\infty E|v_1|^2dt - \frac{1}{\gamma}\int_0^\infty E\{y^*(B_1^*H_0B_1 + B_2^*H_0B_2)y\}\,dt + \frac{1}{2\pi}\int_{-\infty}^{\infty}E|v_{1F}(i\omega)|^2\,\mathrm{Re}\,2\varkappa_0(i\omega)Q_1\,d\omega$$

(In /10/ an Ito system with deterministic control is considered. In our case, $v_{1F}(i\omega)$ fundamentally depends on the processes w_1, w_2, which rules out direct application of the technique of /10/).

Constructing from (4.3) the quadratic form F and substituting $v_1 = v - \mu\sigma$, we obtain

$$F(y, v) = y^* R_1 y + 2y^* Q_1 v_1 + G_1 |v_1|^2 + \varepsilon |v|^2 \equiv F_2(y, v_1) + \varepsilon |v|^2$$

where ε is the number from (4.10), the matrices R, Q_1, G_1 are defined by (4.5). From the preceding bounds and condition (4.10), we obtain

$$\int_0^\infty EF_2(y, v_1)\,dt \geqslant 0, \quad \forall v_1 \in V$$

Therefore, condition (2.5) of Theorem 1 is satisfied. By Theorem 1, controller (1.5) stabilizes system (4.1), (4.2).

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